

LOCAL COMPARABILITY OF MEASURES, AVERAGING AND MAXIMAL AVERAGING OPERATORS

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ABSTRACT. We explore the consequences for the boundedness properties of averaging and maximal averaging operators, of the following local comparability condition for measures: Intersecting balls of the same radius have comparable sizes. Since in geometrically doubling spaces this property yields the same results as doubling, we study under which circumstances it is equivalent to the latter condition, and when it is more general. We also study the concrete case of the standard gaussian measure, where this property fails, but nevertheless averaging operators are uniformly bounded, with respect to the radius, in L^1 . However, such bounds grow exponentially fast with the dimension, very much unlike the case of Lebesgue measure.

1. INTRODUCTION

In the development of analysis in metric spaces, the doubling condition on a measure has played a considerable role, cf. for instance [He], [HKST] and the references contained therein. However, the existence of a doubling measure imposes severe restrictions on the growth of the spaces under consideration: They must be geometrically doubling (of homogeneous type in the terminology of [CoWe1], cf. Definition 2.7 below; we always assume that measures are not identically 0). This excludes many spaces of interest such as, for instance, hyperbolic spaces, as well as several other manifolds with negative curvature.

Hence, there have been efforts to remove or at least weaken the doubling condition whenever possible. A considerable amount of work has been made in this area, regarding singular integrals and Calderón-Zygmund operators cf., for instance, [To] and the references contained therein. Here we are interested in the boundedness properties of the Hardy-Littlewood maximal function defined by a Borel measure μ . Good boundedness results appear to be related to the following property, studied in this paper: Intersecting balls with the same radius have a comparable size (cf. Definition 3.1 below for the precise statement). Since this hypothesis does not apply (at least, not directly) to balls that fail to intersect, we shall say that μ satisfies a *local* comparability condition, even though it applies to balls both large and small, and not just to small balls.

The idea of using local comparability instead of doubling is suggested by [NaTa], where this “uniformity condition” (as is called there, cf. p. 737) is combined with the notion of microdoubling to define “strong microdoubling”. But local comparability by itself is worthy

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of study, independently of any microdoubling conditions. In geometrically doubling spaces, this more precise hypothesis yields the same results as doubling, and is sometimes equivalent to it. But in general, it is satisfied by a wider class of measures.

A second source of interest comes from attempts to understand which factors influence the size of bounds for averaging and maximal averaging operators. This leads us to consider measures for which local comparability is missing. But even for a doubling measure, if one is interested in quantitative aspects of the bounds, one may want to keep track of the local comparability constant, which often will be much smaller than the doubling constant (for instance, 1 and 2^d for d -dimensional Lebesgue measure).

This paper is organized as follows. In Section 2 we review some standard definitions and facts, introducing also the terminology of blossoms. The r -blossom of a set is the just its r -neighborhood, but we find the new terminology more convenient when talking about properties of measures. In particular, to carry out the usual Vitali covering argument, instead of doubling it is enough to assume that the measure blossoms boundedly (cf. Definition 2.4 below).

It is natural to ask which properties of the measure determine, or have an influence, in the behaviour of the bounds satisfied by averaging and maximal averaging operators. In this regard, Section 3 contains the definition of local comparability, and Section 4 considers averaging operators when local comparability is missing. In general metric spaces, without local comparability averaging operators may fail to be bounded for all $p < \infty$. Nevertheless, in the special case of \mathbb{R}^d with the standard gaussian measure, we show that $\sup_{r>0} \|A_r\|_{L^1 \rightarrow L^1} \leq (2 + \varepsilon)^d$ for every $\varepsilon > 0$ and d large enough. However, lack of local comparability makes itself felt in the fact that these bounds grow exponentially fast with the dimension, for all $1 \leq p < \infty$. Using an argument of A. Criado and P. Sjögren, we show that for every p in $[1, \infty)$ and every d sufficiently large, the weak type (p, p) constants satisfy $\|A_{\frac{\sqrt{3d-3}}{2}}\|_{L^p \rightarrow L^{p,\infty}} > 1.019^{d/p}$.

In the case of Lebesgue measure in \mathbb{R}^d , E. M. Stein showed that for the centered maximal function M associated to euclidean balls, the best strong type (p, p) bounds are independent of d , and hence of the doubling constant 2^d ([St1], [St2], [StSt], see also [St3]). In fact, for $p \geq 2$, P. Auscher and M. J. Carro gave the explicit bound $\|M\|_{L^p \rightarrow L^p} \leq (2 + \sqrt{2})^{2/p}$ ([AuCa]). Comparing the situation with the gaussian measure, we see that taking the supremum over radii can have a much smaller impact on the size of constants than considering measures without local comparability.

Stein's result was generalized to the maximal function defined using an arbitrary norm by J. Bourgain ([Bou1], [Bou2]) and A. Carbery ([Ca]) when $p > 3/2$. For ℓ_q balls, $1 \leq q < \infty$, D. Müller [Mu] showed that uniform bounds again hold for every $p > 1$ (given $1 \leq q < \infty$, the ℓ_q balls are defined using the norm $\|x\|_q := (|x_1|^q + |x_2|^q + \dots + |x_d|^q)^{1/q}$). Finally, in [Bou3], Bourgain proved that for cubes (balls with respect to the ℓ_∞ norm) the uniform bounds hold for every $p > 1$. Since for cubes it is known that the weak type $(1, 1)$ constants diverge to infinity (cf. [A], and for the highest lower bounds currently known, cf. [IaSt]) this is the only case where a fairly complete picture is available. Now if the local comparability constant is a key factor here, then one would expect that maximal functions defined using

Lebesgue measure and different balls, would all behave in a similar way, both regarding weak and strong type constants. But I have made no progress in this direction.

Since in geometrically doubling metric spaces local comparability yields the same boundedness results as doubling, it is natural to enquire to what extent the first property is more general than the second. This is done in Section 4, where among other results, it is proven that for geometrically doubling spaces that are quasiconvex, or have the approximate midpoint property, or where all the balls are connected, local comparability is equivalent to doubling. In particular, this is the case for euclidean spaces. However, we shall also see that there is an arc-connected, geometrically doubling metric space, with a non-doubling measure satisfying a local comparability condition. Generally speaking, in spaces with poor connectivity properties, or with large gaps (as is often the case, for instance, with fractals), or where the “intrinsic” and “ambient” metrics are not comparable, the two classes of measures can be quite different.

While the existence of a doubling measure imposes restrictions on the growth of the space, this is not the case with local comparability, which is just a uniformity condition. It may well be that local comparability can yield positive results beyond geometrically doubling spaces, but I have made no progress in this direction.

2. NOTATION AND BACKGROUND MATERIAL

We will use $B(x, r) := \{y \in X : d(x, y) < r\}$ to denote open balls, $\overline{B(x, r)}$ to denote their topological closures, and $B^{cl}(x, r) := \{y \in X : d(x, y) \leq r\}$ to refer to closed balls (consider $\overline{B(0, 1)}$ in \mathbb{Z} to see the difference).

Definition 2.1. We say that (X, d, μ) is a *metric measure space* if μ is a Borel measure on the metric space (X, d) , such that for all balls $B(x, r)$, $\mu(B(x, r)) < \infty$, and furthermore, μ is τ -smooth. A Borel measure is τ -smooth if for every collection $\{U_\alpha : \alpha \in \Lambda\}$ of open sets, $\mu(\cup_\alpha U_\alpha) = \sup \mu(\cup_{i=1}^n U_{\alpha_i})$, where the supremum is taken over all finite subcollections of $\{U_\alpha : \alpha \in \Lambda\}$.

In separable metric spaces, arbitrary unions of open sets can be reduced to countable unions, so τ -smoothness is an immediate consequence of countable additivity; trivially also, all Radon measures are τ -smooth. The hypothesis of τ -smoothness is rather weak, since it is consistent with standard set theory (Zermelo-Fraenkel with Choice) that in every metric space, every Borel measure which assigns finite measure to balls is τ -smooth (cf. [Fre, Theorem (a), pg. 59]). Thus, in standard mathematical practice we will never encounter an example where X is metric and μ fails to be τ -smooth.

Definition 2.2. Let (X, d, μ) be a metric measure space and let g be a locally integrable function on X . For each fixed $r > 0$ and each $x \in X$ such that $0 < \mu(B(x, r))$, the averaging operator $A_{r, \mu}$ is defined as

$$(1) \quad A_{r, \mu} g(x) := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} g d\mu.$$

In addition, the centered Hardy-Littlewood maximal operator M_μ is given by

$$(2) \quad M_\mu g(x) := \sup_{\{r>0: \mu B(x,r)>0\}} A_{r,\mu} |g|(x),$$

while the uncentered Hardy-Littlewood maximal operator M_μ^u is defined via

$$(3) \quad M_\mu^u g(x) := \sup_{\{r>0, y \in X: d(x,y)<r \text{ and } \mu B(y,r)>0\}} A_{r,\mu} |g|(y).$$

According to our convention, averaging operators are defined almost everywhere (since by τ -smoothness the complement of the support has measure zero) while maximal operators are defined everywhere, for given any $x \in X$ there exists an $r > 0$ such that $\mu B(x, r) > 0$. Also, maximal operators can be defined using closed balls instead of open balls, and this does not change their values, since open balls can be approximated from within by closed balls, and closed balls can be approximated from without by open balls. When the measure is understood, we will omit the subscript μ from $A_{r,\mu}$, M_μ , and M_μ^u .

For a given p with $1 \leq p < \infty$, an operator T satisfies a weak type (p, p) inequality if there exists a constant $c > 0$ such that

$$(4) \quad \mu(\{Tg \geq \alpha\}) \leq \left(\frac{c \|g\|_{L^p(\mu)}}{\alpha} \right)^p,$$

where $c = c(p, \mu)$ depends neither on $g \in L^p(\mu)$ nor on $\alpha > 0$. The lowest constant c that satisfies the preceding inequality is denoted by $\|T\|_{L^p \rightarrow L^p, \infty}$. Likewise, if there exists a constant $c > 0$ such that

$$(5) \quad \|Tg\|_{L^p(\mu)} \leq c \|g\|_{L^p(\mu)},$$

we say that T satisfies a strong type (p, p) inequality. The lowest such constant (the operator norm) is denoted by $\|T\|_{L^p \rightarrow L^p}$.

Definition 2.3. A Borel measure μ on (X, d) is *doubling* if there exists a $C > 0$ such that for all $r > 0$ and all $x \in X$, $\mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$.

The following definition comes essentially from [NaTa, p. 739], but the terminology is new.

Definition 2.4. Given a set S we define its *s-blossom* as the enlarged set

$$(6) \quad Bl(S, s) := \cup_{x \in S} B(x, s),$$

and its *uncentered s-blossom* as the set

$$(7) \quad Blu(S, s) := \cup_{x \in S} \cup \{B(y, s) : x \in B(y, s)\}.$$

When $S = B(x, r)$, we simplify the notation and write $Bl(x, r, s)$, instead of $Bl(B(x, r), s)$, and likewise for uncentered blossoms. In the latter case, we allow $r = 0$:

$$(8) \quad Blu(x, 0, s) := \cup \{B(y, s) : x \in B(y, s)\}.$$

We say that μ *blossoms boundedly* if there exists a $K \geq 1$ such that for all $r > 0$ and all $x \in X$, $\mu(Blu(x, r, r)) \leq K\mu(B(x, r)) < \infty$.

Remark 2.5. Note that $Bl(S, s)$ is just the s -neighborhood of S , and the uncentered blossom is just an abbreviation for the iterated blossom: $Blu(S, s) = Bl(Bl(S, s), s)$. The introduction of the new notation is motivated by the fact that often we will be blossoming balls $B(x, r)$, and it will be convenient to keep track both of r and s .

Remark 2.6. Blossoms can be defined using closed instead of open balls, in an entirely analogous way. We mention that in the euclidean case, and more generally, in spaces with the approximate midpoint property (cf. Definition 5.6) there is no difference between $Bl(x, r, r)$ and $B(x, 2r)$, nor between $Blu(x, r, r)$ and $B(x, 3r)$, see [A2, Theorem 2.13]. But in general, balls are larger.

Geometrically doubling spaces have received many names. In the book [CoWe1], these spaces are called (in french) spaces of homogenous type. However, in the paper [CoWe2], the authors switched notation and started calling spaces of homogenous type to those endowed with a doubling measure, after which this became the more common terminology. Both kinds of spaces (geometrically doubling and with a doubling measure) have also been called “doubling spaces”, which is why I am avoiding this expression.

Definition 2.7. A metric space is *D -geometrically doubling* if there exists a positive integer D such that every ball of radius r can be covered with no more than D balls of radius $r/2$.

Of course, metric spaces endowed with a doubling measure, and geometrically doubling spaces, are closely related, the latter being a generalization of the former. If μ on X is doubling, then X is geometrically doubling, cf. [CoWe1, Remarque, p. 67] (but this is not necessarily the case for measures that blossom boundedly, see Theorem 6.1 below). For a trivial example of a geometrically doubling space which does not carry any doubling measure, just consider \mathbb{Q} . For a less trivial example, there are open subsets of \mathbb{R} which do not carry any doubling measures (cf. [He, Remark 13.20 (d)]). But if the geometrically doubling space is complete, then a doubling measure can be defined on it (cf. [LuSa]).

Arguments in analysis that rely on covering theorems often extract a disjoint collection from the original cover, in such a way that not too much measure is disregarded. Now the doubling condition gives us control on the size of all balls contained in $B(x, 2r)$, regardless of whether they intersect $B(x, r)$ or not. Since to disjointify we only need to consider balls that do intersect $B(x, r)$, it is advantageous to use $Bl(x, r, r)$ and $Blu(x, r, r)$ instead of $B(x, 2r)$ and $B(x, 3r)$. The idea of using blossoms can be found in [Li], for locally compact amenable groups (where in principle there are no balls); and in the metric setting, it comes from [NaTa]. Next we rewrite, for the reader’s convenience, a well known argument, using the terminology of blossoms.

Given a ball $B(x_i, r_i)$ we shall sometimes use B_i as an abbreviation. It is understood not only that $B_i = B(x_i, r_i)$ as a set, but also that x_i and r_i are a selected center and a selected radius of B_i (recall that in arbitrary metric spaces, neither the center nor the radius of a ball are in general unique). In particular, it might happen that $B_i = B_j$ for some pair $i \neq j$.

When a measure blossoms boundedly, the following version of the Vitali covering lemma holds.

Theorem 2.8. Vitali covering lemma. *Let (X, d, μ) be a metric measure space. Assume there exists a constant $K \geq 1$ such that for every $x \in X$ and every $r > 0$, $\mu(\text{Blu}(x, r, r)) \leq K \mu(B(x, r))$. Then, for every finite collection of balls $B(x_1, r_1), \dots, B(x_n, r_n)$, there exists a disjoint subcollection $B(x_{i_1}, r_{i_1}), \dots, B(x_{i_m}, r_{i_m})$ with*

$$(9) \quad \mu(\cup_{i=1}^n B(x_i, r_i)) \leq K \mu(\cup_{j=1}^m B(x_{i_j}, r_{i_j})).$$

Proof. We may assume that the original collection, which we will abbreviate by B_1, \dots, B_n , is ordered by decreasing radii. Let $B_{i_1} := B_1$, select B_{i_2} to be the first ball in the list not intersecting B_{i_1} , and in general, choose B_{i_k} as the first ball in the list not intersecting any of the previously selected balls. Then the process finishes after a finite number of steps with, let's say, the ball B_{i_m} .

We need to control the mass lost with the balls not chosen. Let $B_{j_1}^1, \dots, B_{i_k}^1$ be the collection of all balls intersecting B_{i_1} . Then $\cup_{s=1}^k B_{i_s}^1 \subset \text{Blu}(x_1, r_1, r_1)$, so $\mu(\cup_{s=1}^k B_{i_s}^1) \leq \mu(\text{Blu}(x_1, r_1, r_1)) \leq K \mu(B(x_1, r_1))$. Repeating this argument with the other balls, we obtain (9). \square

The preceding theorem entails the weak type (1,1) of the maximal operator, and by interpolation, the corresponding strong type bounds. One also has the Lebesgue Differentiation Theorem for measures that blossom boundedly, as in the case of doubling measures.

Regarding the strong type bounds, we mention that once an averaging or maximal averaging operator is bounded in L^r for some $r > 0$, it is bounded in L^p for all $p > r$, with operator norm that approaches 1 (something that is not always observed in published results). There is no need to use Riesz-Thorin (for positive sublinear operators) to obtain this, it immediately follows from Jensen's inequality.

Theorem 2.9. *Let (X, \mathcal{A}, μ) be a measure space, let $0 < r < \infty$, and let T be an averaging or maximal averaging operator, bounded on $L^r(X, \mu)$ and with operator norm c_r . Then T is bounded on L^p for all $p \geq r$, with operator norm $c_p \leq c_r^{r/p}$.*

Proof. For all p such that $r < p < \infty$, and all $f \in L^p$, $f \geq 0$, we have $f^{p/r} \in L^r$, so by Jensen's inequality,

$$\int [T(f)]^p d\mu \leq \int [T(f^{p/r})]^r d\mu \leq c_r^r \int f^p d\mu,$$

and the result follows by taking p -th roots. \square

3. LOCAL COMPARABILITY

Definition 3.1. We say that a measure μ satisfies a *local comparability condition* for the radius r if there exists a constant $C \in [1, \infty)$ such that for all pairs of points $x, y \in X$, whenever $d(x, y) < r$, we have

$$\mu(B(x, r)) \leq C \mu(B(y, r)).$$

If the constant C can be chosen to be independent of r , then we say that μ satisfies a *C local comparability condition*. We denote the smallest such C by $C(\mu)$ or C_μ .

Interchanging x and y in the preceding definition leads to

$$\frac{1}{C} \leq \frac{\mu(B(x, r))}{\mu(B(y, r))} \leq C,$$

provided $\mu(B(x, r)) > 0$ (of course, $\mu(B(x, r)) = 0$ if and only if $\mu(B(y, r)) = 0$). While it is always possible to assume that μ has full support, by disregarding, if needed, a measure zero set, this can lead to substantial changes in the geometry of the resulting space, since many properties are not inherited by subsets. So even though we will always suppose that μ is not identically 0, full support will not be assumed.

Example 3.2. Suppose (X, d) is an ultrametric space (so the triangle inequality is replaced by the stronger condition $d(x, y) \leq \max\{d(x, z), d(z, y)\}$). It follows that $B(x, r) = B(y, r)$ whenever $d(x, y) < r$, so for every measure μ on X , the local comparability condition is trivially satisfied, with $C(\mu) = 1$.

In order to use uncentered blossoms, the following obvious estimate is useful.

Lemma 3.3. *Let (X, d, μ) be a metric measure space, and let μ satisfy a C local comparability condition. If $B(x, r) \cap B(y, r) \neq \emptyset$, then $\mu(B(x, r)) \leq C_\mu^2 \mu(B(y, r))$.*

Proof. Let $z \in B(x, r) \cap B(y, r)$. Since $d(x, z) < r$ and $d(z, y) < r$, we have that

$$\mu(B(x, r)) \leq C_\mu \mu(B(z, r)) \leq C_\mu^2 \mu(B(y, r)).$$

□

The local comparability condition can be equivalently stated in terms of closed balls.

Lemma 3.4. *A measure μ satisfies a C local comparability condition if and only if for every $r > 0$ and all pairs of points $x, y \in X$, whenever $d(x, y) \leq r$, we have*

$$\mu(B^{cl}(x, r)) \leq C\mu(B^{cl}(y, r)).$$

Proof. Suppose μ satisfies a C local comparability condition, and let $d(x, y) \leq r$. Then for every $n \geq 1$, $\mu(B(x, r + n^{-1})) \leq C\mu(B(y, r + n^{-1}))$. Taking the limit as $n \rightarrow \infty$ we obtain $\mu(B^{cl}(x, r)) \leq C\mu(B^{cl}(y, r))$. For the other direction, suppose that $0 < d(x, y) < r$, select $N \gg 1$ such that $d(x, y) \leq r - N^{-1}$, and use $\mu(B^{cl}(x, r - n^{-1})) \leq C\mu(B^{cl}(y, r - n^{-1}))$ whenever $n \geq N$. Letting $n \rightarrow \infty$ we obtain $\mu(B(x, r)) \leq C\mu(B(y, r))$. □

It is easy to see (and we prove it below) that in geometrically doubling spaces local comparability implies boundedness of blossoms. Thus, it is interesting to see how close, or how far away, is local comparability from doubling.

But before we do so, we consider the behavior of averaging operators when local comparability is missing, with special emphasis in the case of the standard Gaussian measure.

4. AVERAGING OPERATORS WITHOUT LOCAL COMPARABILITY

As noted in [NaTa, p. 737], if μ satisfies a C local comparability condition, then, a simple application of Fubini's Theorem yields, for all averaging operators A_r , $r > 0$, the uniform bound $\|A_r\|_{L^1 \rightarrow L^1} \leq C$. We recall the argument: Suppose that for a fixed radius s , and all $x, y \in X$ with $0 < d(x, y) < s$, we have $\mu(B(x, s)) \leq C\mu(B(y, s))$. If $0 \leq f \in L^1(\mu)$, then

$$(10) \quad \|A_s f\|_{L^1} = \int_X \int_X \frac{\mathbf{1}_{B(x,s)}(y)}{\mu B(x, s)} f(y) d\mu(y) d\mu(x)$$

$$(11) \quad = \int_X f(y) \int_X \frac{\mathbf{1}_{B(y,s)}(x)}{\mu B(x, s)} d\mu(x) d\mu(y) \leq C \int_X f(y) d\mu(y),$$

so $\|A_s\|_{L^1 \rightarrow L^1} \leq C$, and hence local comparability for s entails the bound C for the corresponding averaging operator, while local comparability (for all radii) entails uniform bounds for all the averaging operators.

Example 4.1. The bound $\|A_s\|_{L^p \rightarrow L^p} \leq C$, for a specific radius s , may fail for all $p \in [1, \infty)$, if μ lacks local comparability for balls of radius s . We define next a path connected subset $B \subset \mathbb{R}^2$, on which we use the path metric d instead of the ambient space metric: The distance between two points is the length of the shortest path joining them. Start with the positive x -axis. For each $n \geq 1$, select n points from the circumference of radius 1 centered at $(3n, 0)$: $z_{n,1}, \dots, z_{n,n} \in \{(x, y) \in \mathbb{R}^2 | (x - 3n)^2 + y^2 = 1\}$. Join the points $z_{n,1}, \dots, z_{n,n}$ to the center $(3n, 0)$ using straight line segments (radii), let B be the union of the positive x -axis with all the “spikes” attached to the centers $(3n, 0)$, and let μ be the counting measure on the points $(3n, 0)$ and $z_{n,1}, \dots, z_{n,n}$, for every $n \geq 1$ (clearly, B is separable, and all balls have finite measure). Now $d(z_{n,k}, (3n, 0)) = 1$, while if $m \neq n$, $d(z_{n,k}, z_{m,j}) \geq 5$. Thus, $\mu B(z_{n,k}, 3/2) = 2$ and $\mu B((3n, 0), 3/2) = n + 1$, so local comparability fails for $s = 3/2$. Let $f_n = \mathbf{1}_{\{(3n, 0)\}}$, and fix $n, p \geq 1$ with $n \gg 2^p$. Then $(A_{3/2} \mathbf{1}_{\{(3n, 0)\}})(z_{n,k}) = 1/2$, from whence it follows that $\|A_{3/2}\|_{L^p \rightarrow L^p}^p \geq n2^{-p}$. For the same reason (or by interpolation) $A_{3/2}$ satisfies no weak (p,p) type bounds.

It is nevertheless possible to have uniform L^p bounds for A_r without local comparability. For instance, in \mathbb{R}^d , because of the Besicovitch covering theorem, the centered maximal function defined by an arbitrary measure is of weak type (1,1), and by interpolation, bounded on L^p for all $1 < p < \infty$ (with bounds that grow exponentially with the dimension). Thus, averaging operators satisfy L^p bounds independent of r , for $p > 1$. We shall see later that exponential growth with d can actually happen for A_s with suitably chosen s .

Regarding L^1 bounds, it may happen, even in the absence of local comparability, that the term

$$\int_X \frac{\mathbf{1}_{B(y,s)}(x)}{\mu B(x, s)} d\mu(x)$$

in the left hand side of (11) can still be controled, if the ratio $\frac{\mu B(y,s)}{\mu B(x,s)}$ becomes large on sets of sufficiently small measure. Next we consider the standard exponential distribution in one

dimension, and the standard gaussian measure in all dimensions. By way of comparison with the results that follow, recall that for Lebesgue measure in \mathbb{R}^d , for every $r > 0$ and $d \geq 1$, $\|A_r\|_{L^1 \rightarrow L^1} = 1$ (using (10)-(11)), $\|A_r\|_{L^\infty \rightarrow L^\infty} \leq 1$ (this is obvious), and $\|A_r\|_{L^p \rightarrow L^p} \leq 1$ for $1 < p < \infty$, by interpolation or by Theorem 2.9.

Theorem 4.2. *Consider $\Omega = (0, \infty)$ with the standard exponential distribution, given by $dP(t) = e^{-t}dt$. Then P satisfies a local comparability condition for each radius r , with optimal $C(r) \in [e^r, \max\{2, e^r\}]$; thus, it fails to satisfy a local comparability condition. However, the averaging operators are uniformly bounded, with $1.27 < \|A_1\|_{L^1 \rightarrow L^1} \leq \sup_{r>0} \|A_r\|_{L^1 \rightarrow L^1} \leq 2$.*

Proof. For convenience we extend P to \mathbb{R} by setting $P(-\infty, 0] = 0$. Fix $r > 0$. First we check that for every $x, y > 0$ such that $|x - y| \leq r$, $P((x - r, x + r)) \leq \max\{2, e^r\}P((y - r, y + r))$, and the bound $\max\{2, e^r\}$ cannot be lowered. We may assume, without loss of generality, that $x < y$, which leads to the consideration of the following three cases: $x \geq r$, $x < r \leq y$, and $y < r$. In the first case, $P((x - r, x + r)) \geq P((y - r, y + r))$, and a computation shows that $P((x - r, x + r))/P((y - r, y + r)) \leq e^r$, with equality when $y = x + r$. In the third case, $P((0, x + r)) \leq P((0, y + r))$, and $P((0, y + r))/P((0, x + r)) \leq (1 - e^{-2r})/(1 - e^{-r}) \leq 2$, since for any decreasing function $h \geq 0$, we have $\int_0^{2r} h \leq 2 \int_0^r h$, and furthermore $\lim_{r \rightarrow 0} (1 - e^{-2r})/(1 - e^{-r}) = 2$, so we get arbitrarily close to 2 by letting $x \rightarrow 0, y \rightarrow r$, and then $r \rightarrow 0$. Finally, when $x < r \leq y$, we have $e^{-r} = (1 - e^{-2r})/(e^r - e^{-r}) \leq P((y - r, y + r))/P((x - r, x + r)) \leq (1 - e^{-2r})/(1 - e^{-r}) \leq 2$.

Next, note that for every $w \in (0, \infty)$ and every $r > 0$, $P(B(w, r)) = \int_{\max\{0, w-r\}}^{w+r} e^{-t}dt$, so if $w \geq r$, by the convexity of e^{-t} we have

$$(12) \quad P(B(w, r)) \geq e^{-w}2r,$$

while if $w < r$, then $P(B(w, r)) = \int_0^{w+r} e^{-t}dt = 1 - e^{-r-w} \geq 1 - e^{-r}$. In order to bound

$$(13) \quad \int_0^\infty f(y) \int_0^\infty \frac{\mathbf{1}_{B(y,r)}(x)}{P(B(x,r))} dP(x) dP(y),$$

we break up the outer integral into $\int_0^\infty = \int_0^{2r} + \int_{2r}^\infty$. On the region where $y \geq 2r$, since $|x - y| \leq r$, both $B(x, r) = (x - r, x + r)$ and $B(y, r) = (y - r, y + r)$ so

$$\int_0^\infty \frac{\mathbf{1}_{B(y,r)}(x)}{P(B(x,r))} dP(x) = \int_{y-r}^{y+r} \frac{e^{-x}}{P((x-r, x+r))} dx \leq \int_{y-r}^{y+r} \frac{dx}{2r} dx = 1.$$

If $0 < y < 2r$,

$$\int_0^\infty \frac{\mathbf{1}_{B(y,r)}(x)}{P(B(x,r))} dP(x) = \int_{\max\{0, y-r\}}^{y+r} \frac{e^{-x}}{P((x-r, x+r))} dx = \int_{\max\{0, y-r\}}^r + \int_r^{y+r}.$$

Now

$$\int_{\max\{0, y-r\}}^r \frac{e^{-x}}{P((x-r, x+r))} dx \leq \int_{\max\{0, y-r\}}^r \frac{e^{-x}}{1 - e^{-r}} dx = \frac{e^{-\max\{0, y-r\}} - e^{-r}}{1 - e^{-r}} \leq \frac{1 - e^{-r}}{1 - e^{-r}} = 1,$$

while

$$\int_r^{y+r} \frac{e^{-x}}{P((x-r, x+r))} dx \leq \int_r^{y+r} \frac{dx}{2r} \leq 1,$$

by (12). Hence,

$$(14) \quad \int_0^\infty f(y) \int_0^\infty \frac{\mathbf{1}_{B(y,r)}(x)}{P(B(x,r))} dP(x) dP(y) = 2 \int_0^{2r} f(y) dP(y) + \int_{2r}^\infty f(y) dP(y) \leq 2\|f\|_1.$$

Next, take $r = 1$. We show that $1.27 < \|A_1\|_{L^1 \rightarrow L^1}$. Recall that when the centered maximal operator acts on a measure ν , it is defined via

$$M_\mu \nu(x) := \sup_{\{r>0: \mu(B(x,r))>0\}} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

By a standard approximation argument, instead of a function, we consider a Dirac delta placed at 1. If $x \in (0, 1)$, then

$$A_1 \delta_1(x) = \frac{1}{1 - e^{-1-x}},$$

while if $x \in [1, 2)$, then

$$A_1 \delta_1(x) = \frac{e^x}{e - e^{-1}}.$$

Using the change of variables $u = e^x$ and integrating explicitly, we obtain

$$\begin{aligned} \|A_1\|_{L^1(P) \rightarrow L^1(P)} &\geq \|A_1 \delta_1\|_{L^1(P)} = \int_0^2 A_1 \delta_1(x) e^{-x} dx = \int_0^1 \frac{dx}{e^x - e^{-1}} + \int_1^2 \frac{dx}{e - e^{-1}} \\ &= e \log \left(\frac{e - e^{-1}}{1 - e^{-1}} \right) - e + \frac{1}{e - e^{-1}} > 1.27. \end{aligned}$$

□

Next we consider the case of the standard Gaussian measure γ_d in \mathbb{R}^d . Here local comparability fails for every single radius $r > 0$, but nevertheless, the averaging operators A_r defined by γ_d satisfy uniform L^1 -bounds (in r). We use $\|x\|_2 := (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2}$ to denote the euclidean distance in \mathbb{R}^d . Recall that the standard gaussian measure is given by

$$d\gamma^d(x) = \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{d/2}} dx.$$

Theorem 4.3. *Let (Ω, d, P) be $(\mathbb{R}^d, \|x\|_2, \gamma^d)$, where γ^d is the standard gaussian measure. Given any $r > 0$, γ^d does not satisfy a local comparability condition for r . However, for all $p \geq 1$,*

$$\sup_{r>0} \|A_r\|_{L^p \rightarrow L^p} \leq \left(2^{d-1} \sqrt{2\pi d} + \sqrt{\pi(d+1)} \left(\frac{2}{\sqrt{3}} \right)^{d+1} \right)^{1/p}.$$

Thus, for every $\varepsilon > 0$ and every d large enough,

$$\sup_{r>0} \|A_r\|_{L^p \rightarrow L^p} \leq (2 + \varepsilon)^{d/p}.$$

Furthermore, for every p in $[1, \infty)$ and every d sufficiently large, the weak type (p, p) constants satisfy $\left\| A_{\frac{\sqrt{3d-3}}{2}} \right\|_{L^p \rightarrow L^{p,\infty}} > 1.019^{d/p}$.

Of course, for $p > 1$ the fact that the operators A_r are uniformly bounded in L^p follows from the corresponding bounds for the maximal operators. But the boundedness of A_r for $p = 1$ in the preceding result is new, and for p close to 1, the bounds obtained by interpolation do not blow up when $p \rightarrow 1$, unlike the case of the maximal function inequalities.

The next obvious lemma tells us that the normalizing constants $(2\pi)^{-d/2}$ in the probabilities can be omitted from certain formulas.

Lemma 4.4. *Given a measure μ and a constant $c > 0$, for all $r > 0$, all $f \in L^1$, and all $x \in X$, we have $A_{r,\mu}f(x) = A_{r,c\mu}f(x)$, and $\|A_{r,\mu}\|_{L^1(\mu) \rightarrow L^1(\mu)} = \|A_{r,c\mu}\|_{L^1(c\mu) \rightarrow L^1(c\mu)}$.*

For the next lemma it will be more convenient to use closed balls. Of course, from the viewpoint of the gaussian measure this makes no difference, since the boundaries of balls have measure zero. Next we introduce some notation. We often omit the center of the sphere $\mathbb{S}^{d-1}(0, r)$ when it is the origin, and if it is the unit sphere ($r = 1$) we also omit the radius. Given a unit vector $v \in \mathbb{R}^d$ and $s \in [0, 1]$, the s spherical cap about v is the set $C(s, v) := \{\theta \in \mathbb{S}^{d-1} : \langle \theta, v \rangle \geq s\}$. Spherical caps are just geodesic balls $B_{\mathbb{S}^{d-1}}^{cl}(x, r)$ in \mathbb{S}^{d-1} . Let $e_1 = (1, 0, \dots, 0)$. Given any angle $r \in (0, \pi/2)$, writing $s = \cos r$, we have

$$(15) \quad B_{\mathbb{S}^{d-1}}^{cl}(e_1, r) = C(s, e_1).$$

In particular, $B_{\mathbb{S}^{d-1}}^{cl}(e_1, \pi/6) = C(\sqrt{3}/2, e_1)$.

Lemma 4.5. *Let μ be a rotationally invariant measure on \mathbb{R}^d , let $r > 0$, and let $\|v\|_2 = r$. Then $\mu(B^{cl}(0, r)) \leq 2^{d-1} \sqrt{2\pi d} \mu(B^{cl}(v, r))$.*

Proof. First we recall a well known volumetric argument giving upper bounds on the size of r -nets on the r -sphere. Since for this part we only use the normalized area σ_N^{d-1} on the sphere $\mathbb{S}^{d-1}(0, r)$, we can take $r = 1$. Let $\{v_1, \dots, v_M\}$ be a maximal set of unit vectors in \mathbb{R}^d , subject to the condition that for $i \neq j$, $\|v_i - v_j\|_2 \geq 1$. The maximality of $\{v_1, \dots, v_M\}$ entails that $B^{cl}(0, 1) \subset \cup_1^M B^{cl}(v_i, 1)$, as the following argument shows. Suppose $y \in B^{cl}(0, 1) \setminus \cup_1^M B^{cl}(v_i, 1)$. Then $y \neq 0$, because the origin belongs to all the balls under consideration, so $v := y/\|y\|_2$ lies on the unit sphere. Now if $v \in B^{cl}(v_i, 1)$ for some index i , then $y \in B^{cl}(v_i, 1)$ by the convexity of the ball, so $v \in B^{cl}(0, 1) \setminus \cup_1^M B^{cl}(v_i, 1)$. But then $\|v_i - v\|_2 > 1$ for $i = 1, \dots, M$, contradicting the maximality of $\{v_1, \dots, v_M\}$.

Next, note that all balls $B^{cl}(v_i, 1/2)$ have disjoint interiors, so their radial projections (from the origin) into the unit sphere \mathbb{S}^{d-1} also have disjoint interiors. By rotational invariance we may assume that $v_1 = e_1$. Now the tangent lines to $B^{cl}(e_1, 1/2)$ starting from the origin, form an angle of $\pi/6$ with e_1 , since the radii of $B^{cl}(e_1, 1/2)$ are perpendicular to these tangent lines,

and $\sin(\pi/6) = 1/2$. Thus, the radial projection of $B^{cl}(e_i, 1/2)$ into \mathbb{S}^{d-1} is the geodesic ball, or spherical cap, $B_{\mathbb{S}^{d-1}}^d(e_1, \pi/6) = C(\sqrt{3}/2, e_1)$. By [AlPe, Lemma 2.1],

$$(16) \quad \frac{1}{2^{d-1}\sqrt{2\pi d}} \leq \sigma_N^{d-1}(C(\sqrt{3}/2, e_1)),$$

so $\sum_1^M \sigma_N^{d-1}(C(\sqrt{3}/2, v_i)) = M\sigma_N^{d-1}(C(\sqrt{3}/2, e_1)) \leq 1$, from which it follows that $M \leq 2^{d-1}\sqrt{2\pi d}$.

Let us now return to the original μ and $r > 0$. By invariance under rotations, for all i we have $\mu B^{cl}(rv_i, r) = \mu B^{cl}(rv_1, r)$. Since $B^{cl}(0, r) \subset \cup_1^M B^{cl}(rv_i, r)$, it follows that $\mu B^{cl}(0, r) \leq \mu \cup_1^M B^{cl}(rv_i, r) \leq \sum_1^M \mu B^{cl}(rv_i, r) \leq 2^{d-1}\sqrt{2\pi d} \mu(B^{cl}(rv_1, r))$. \square

Corollary 4.6. *Let μ be a radial, radially decreasing measure on \mathbb{R}^d . That is, $d\mu(x) = f(x)dx$, where f is a locally integrable, radial and radially decreasing function. Then for every $g \in L^1(\mu)$, $\|\mathbf{1}_{B(0,r)} A_{r,\mu} g\|_{L^1(\mu)} \leq 2^{d-1}\sqrt{2\pi d} \|\mathbf{1}_{B(0,2r)} g\|_{L^1(\mu)}$, where $A_{r,\mu}$ is defined using the closed balls $B^{cl}(x, r)$.*

Proof. Note that of all balls $B^{cl}(x, r)$ containing the origin, the ones with smallest measure are those for which $\|x\|_2 = r$, since for every $t \in [0, 1)$, for all $z \in B^{cl}(x, r) \setminus B^{cl}(tx, r)$, and all $y \in B^{cl}(tx, r) \setminus B^{cl}(x, r)$, $f(z) < f(y)$. Now if $x \in B(0, r) \setminus \{0\}$ and $0 \leq g \in L^1(\mu)$, then

$$A_{r,\mu} g(x) \leq \frac{\|\mathbf{1}_{B(0,2r)} g\|_{L^1(\mu)}}{\mu B^{cl}(rx/\|x\|_2, r)},$$

so the result follows from the previous lemma. \square

For the rest of this section, we use μ to denote the non-normalized gaussian $d\mu(x) = e^{-\frac{\|x\|_2^2}{2}}$. Occasionally we will write μ^d to specify the dimension d .

Lemma 4.7. *Let $r > 0$ and let $x \in \mathbb{R}^d \setminus B(0, r)$. Then*

$$\mu B(x, r) \geq \frac{e^{-\frac{\|x\|_2^2}{2}} \lambda^d B(0, r)}{\sqrt{\pi(d+1)}} \left(\frac{\sqrt{3}}{2} \right)^{d+1}.$$

Proof. When needed, we will distinguish between balls in \mathbb{R}^d and \mathbb{R}^{d-1} by writing B^d and B^{d-1} respectively. Using rotational invariance, we may assume that $x = se_1$, with $s \geq r$. Since

$$\mu B(se_1, r) \geq \mu(B(se_1, r) \cap B(0, s)) \geq e^{-s^2/2} \lambda^d (B(se_1, r) \cap B(0, s)),$$

all we need to do is to show that

$$(17) \quad \frac{\lambda^d (B(se_1, r) \cap B(0, s))}{\lambda^d (B(0, r))} \geq \frac{1}{\sqrt{\pi(d+1)}} \left(\frac{\sqrt{3}}{2} \right)^{d+1}.$$

First of all, note that the ratio in the left hand side of the preceding inequality is minimized when $s = r$, so we suppose this is the case. Second, dividing all radii by s and cancelling the

factors s^d , we may take $s = 1$. Now

$$\begin{aligned} \lambda^d(B(0, 1) \cap B(e_1, 1)) &= 2\lambda^d(B^d(0, 1) \cap \{x_1 \geq 2^{-1}\}) \\ &= 2\lambda^{d-1}(B^{d-1}(0, 1)) \int_{1/2}^1 \left(\sqrt{1 - x_1^2}\right)^{d-1} dx_1 \\ &\geq 2\lambda^{d-1}(B^{d-1}(0, 1)) \int_{\pi/6}^{\pi/2} \cos^d t \sin t dt = \frac{2}{d+1} \left(\frac{\sqrt{3}}{2}\right)^{d+1} \lambda^{d-1}(B^{d-1}(0, 1)). \end{aligned}$$

Using $\lambda^d(B^d(0, 1)) = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$, together with the following Gamma function estimate (consequence of the log-convexity of Γ on $(0, \infty)$, cf. Exercise 5, pg. 216 of [Web])

$$(18) \quad \left(\frac{d}{2}\right)^{1/2} \leq \frac{\Gamma(1 + d/2)}{\Gamma(1/2 + d/2)},$$

we get

$$(19) \quad \frac{\lambda^d(B(e_1, 1) \cap B(0, 1))}{\lambda^d(B(0, 1))} \geq \frac{1}{\sqrt{\pi(d+1)}} \left(\frac{\sqrt{3}}{2}\right)^{d+1}.$$

□

Proof of Theorem 4.3. Fix $r > 0$ and take $\|x\|_2 \gg 1+r$. To see that μ does not satisfy a local comparability condition for r , we consider the balls $B(x, r)$ and $B((1+3r/\|x\|_2)x, r)$. Since their centers are at distance $3r$, they are disjoint. However, $B(x, r)$, $B((1+3r/(2\|x\|_2))x, r)$ and $B((1+3r/\|x\|_2)x, r)$ form an intersecting chain of balls of length 3, so applying Lemma 3.3 twice, local comparability for r would imply that the measures of $B(x, r)$ and $B((1+3r/\|x\|_2)x, r)$ are comparable, for every x . However,

$$\lim_{x \rightarrow \infty} \frac{\mu(B((1+3r/\|x\|_2)x, r))}{\mu(B(x, r))} \leq \lim_{x \rightarrow \infty} \frac{e^{-\frac{(\|x\|_2+2r)^2}{2}} \lambda^d B(0, r)}{e^{-\frac{(\|x\|_2+r)^2}{2}} \lambda^d B(0, r)} = 0.$$

In order to prove that $\sup_{r>0} \|A_r\|_{L^1 \rightarrow L^1} \leq (2+\varepsilon)^d$ for d large, we split $A_{r,\mu}$ into $A_{r,\mu} = \mathbf{1}_{B(0,r)} A_{r,\mu} + \mathbf{1}_{B(0,r)^c} A_{r,\mu}$. Let $0 \leq g \in L^1(\mu)$. The bound

$$\|\mathbf{1}_{B(0,r)} A_{r,\mu} g\|_{L^1(\mu)} \leq 2^{d-1} \sqrt{2\pi d} \|\mathbf{1}_{B(0,2r)} g\|_{L^1(\mu)}$$

is a special case of Corollary 4.6, together with the fact that open and closed balls have the same gaussian measure. Regarding the second term,

$$(20) \quad \|\mathbf{1}_{B(0,r)^c} A_{r,\mu} g\|_{L^1(\mu)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{B(x,r)}(y) \mathbf{1}_{B(0,r)^c}(x)}{\mu B(x, r)} g(y) d\mu(y) d\mu(x)$$

$$(21) \quad = \int_{\mathbb{R}^d} g(y) \int_{\mathbb{R}^d} \frac{\mathbf{1}_{B(y,r)}(x) \mathbf{1}_{B(0,r)^c}(x)}{\mu B(x, r)} d\mu(x) d\mu(y).$$

Now fix y . By Lemma 4.7 we have

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{B(y,r)}(x) \mathbf{1}_{B(0,r)^c}(x)}{\mu B(x,r)} d\mu(x) &= \int_{B(0,r)^c \cap B(y,r)} \frac{e^{-\frac{\|x\|_2^2}{2}} dx}{\mu B(x,r)} \\ &\leq \sqrt{\pi(d+1)} \left(\frac{2}{\sqrt{3}} \right)^{d+1} \int_{B(y,r)} \frac{dx}{\lambda^d B(0,r)} = \sqrt{\pi(d+1)} \left(\frac{2}{\sqrt{3}} \right)^{d+1}. \end{aligned}$$

Therefore

$$\begin{aligned} \|A_{r,\mu}g\|_{L^1(\mu)} &= \|\mathbf{1}_{B(0,r)}A_{r,\mu}g\|_{L^1(\mu)} + \|\mathbf{1}_{B(0,r)^c}A_{r,\mu}g\|_{L^1(\mu)} \\ &\leq \left(2^{d-1}\sqrt{2\pi d} + \sqrt{\pi(d+1)} \left(\frac{2}{\sqrt{3}} \right)^{d+1} \right) \|g\|_{L^1(\mu)}, \end{aligned}$$

and from Theorem 2.9 we conclude that for every $p \geq 1$,

$$\|A_{r,\mu}\|_{L^p(\mu) \rightarrow L^p(\mu)} \leq \left(2^{d-1}\sqrt{2\pi d} + \sqrt{\pi(d+1)} \left(\frac{2}{\sqrt{3}} \right)^{d+1} \right)^{1/p}.$$

Regarding the lower bounds for the weak type constants, the argument we use is the same as in [CriSjo], together with gaussian concentration. The basic idea is that since the standard gaussian measure in \mathbb{R}^d behaves essentially as normalized area on the sphere $\mathbb{S}^{d-1}(\sqrt{d})$, centered at 0 and of radius \sqrt{d} , a single well chosen radius is enough to witness the exponential growth of constants with the dimension. So the argument given by Criado and Sjogren for the maximal operator essentially yields the same result for certain averaging operators. Of course, since the standard gaussian measure is not singular, but absolutely continuous, one still has to show that small changes in the center of a ball lead to small changes in the average.

First, we estimate from below the measure of the region bounded between $\mathbb{S}^{d-1}(\sqrt{d-1} - 1/\sqrt{d-1})$ and $\mathbb{S}^{d-1}(\sqrt{d-1})$ (the region between the radii \sqrt{d} and $\sqrt{d-1}$ could be used in an entirely analogous way, but since we want to cite estimates from [CriSjo] directly, rather than to redo them, and they use $\sqrt{d-1}$ as the largest radius, so do we). Denote by

$$\sigma_{d-1}(\mathbb{S}^{d-1}) := \frac{d \pi^{d/2}}{\Gamma(1 + d/2)}$$

the area of the unit sphere \mathbb{S}^{d-1} . In what follows we utilize Stirling's formula, as well as approximations to e , so the inequality below holds for d large enough. Since $g(r) := r^{d-1}e^{-r^2/2}$ is increasing for $0 < r < \sqrt{d-1}$,

$$(22) \quad \int_{B(0,\sqrt{d-1}) \setminus B(0,\sqrt{d-1}-1/\sqrt{d-1})} d\gamma^d(x) = \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{(2\pi)^{d/2}} \int_{\sqrt{d-1}-1/\sqrt{d-1}}^{\sqrt{d-1}} g(r) dr$$

$$(23) \quad \geq \frac{d \pi^{d/2}}{(2\pi)^{d/2} \Gamma(1 + d/2)} g(\sqrt{d-1} - 1/\sqrt{d-1}) \int_{\sqrt{d-1}-1/\sqrt{d-1}}^{\sqrt{d-1}} dr > \frac{1}{(\pi e^3 d)^{1/2}}$$

for d large. We want to obtain a good estimate from below for

$$A_{\frac{\sqrt{3d-3}}{2}} \mathbf{1}_{B(0, \frac{\sqrt{3d-3}}{2})}$$

on the region $D := B(0, \sqrt{d-1}) \setminus B(0, \sqrt{d-1} - 1/\sqrt{d-1})$. Let $1 \leq p < \infty$, let $0 < r, R < 1$, and let $c > 0$ satisfy, for all $\alpha > 0$, the inequality

$$(24) \quad \gamma^d(\{A_{R\sqrt{d-1}, \gamma^d} \mathbf{1}_{B(0, r\sqrt{d-1})} \geq \alpha\}) \leq \left(\frac{c \|\mathbf{1}_{B(0, r\sqrt{d-1})}\|_{L^p(\gamma^d)}}{\alpha} \right)^p,$$

or equivalently

$$(25) \quad \frac{\alpha \gamma^d(\{A_{R\sqrt{d-1}, \gamma^d} \mathbf{1}_{B(0, r\sqrt{d-1})} \geq \alpha\})^{1/p}}{\gamma^d(B(0, r\sqrt{d-1}))^{1/p}} \leq c.$$

To find a (sufficiently high) uniform lower bound α for

$$(26) \quad A_{R\sqrt{d-1}, \gamma^d} \mathbf{1}_{B(0, r\sqrt{d-1})}(x) = \frac{\gamma^d(B(0, r\sqrt{d-1}) \cap (B(x, R\sqrt{d-1})))}{\gamma^d(B(x, R\sqrt{d-1}))},$$

whenever $x \in D$, we use the following facts, taken from [CriSjo, Proof of Lemma 5.1] (note that we have chosen a different, more common normalization for the gaussian measure, but this makes no essential difference; for the justification of the assertions below we refer the reader to the original paper). Criado and Sjögren show that if $\|x\|_2^2 = \sqrt{d-1}$, then for each $R \in (0, 1)$, an $r \in (0, 1)$ can be chosen in such a way that

$$(27) \quad \frac{\gamma^d(B(0, r\sqrt{d-1}) \cap B(x, R\sqrt{d-1}))}{\gamma^d(B(x, R\sqrt{d-1}))} \geq \Theta\left(\frac{1}{\sqrt{d-1}}\right),$$

where Θ denotes exact order. In view of this bound and of the denominator in (25), we want to select r as small as possible, which is where the choice $r = R = \sqrt{3}/2$ comes from, as we shall see next. Let

$$F(t, R) := \left(t - \frac{(1+t-R^2)^2}{4} \right) e^{-t},$$

where $(1-R)^2 \leq t \leq (1+R)^2$, let

$$t(R) := 2 + R^2 - \sqrt{1 + 4R^2},$$

and let

$$G(R) := F(t(R), R).$$

For each fixed R , $t(R)$ maximizes F , so $F(t, R) \leq G(R)$ (cf. [CriSjo, p. 609] for justifications of the choices and claims made). Set $r(R) := \sqrt{t(R)}$. Since $R = \sqrt{3}/2$ is the only zero of t' in $(0, 1)$ and $t''(\sqrt{3}/2) > 0$, the function $t(R)$ has a local minimum there, which is easily seen to be the unique global minimum (for instance, by checking the endpoints). Hence, for $0 < R < 1$, $t(R) \geq t(\sqrt{3}/2) = 3/4$, and we choose $r = \sqrt{3}/2 = R$. Given $x \in D$, by rotational

invariance we may suppose that $x = ue_1$, where $\sqrt{d-1} - 1/\sqrt{d-1} \leq u \leq \sqrt{d-1}$, and as before, e_1 is the first vector in the standard basis of \mathbb{R}^d . Criado and Sjögren show that

$$(28) \quad \gamma^d B(\sqrt{d-1} e_1, R\sqrt{d-1}) \leq \frac{2\sigma_{d-2}(\mathbb{S}^{d-2}) (\sqrt{d-1})^d R}{(d-1)\sqrt{1-R^2}} G(R)^{\frac{d-1}{2}} =: V(R).$$

Using this, for $ue_1 \in D$ we have

$$(29) \quad \frac{\gamma^d(B(0, \frac{\sqrt{3}}{2}\sqrt{d-1}) \cap B(ue_1, \frac{\sqrt{3}}{2}\sqrt{d-1}))}{\gamma^d(B(ue_1, \frac{\sqrt{3}}{2}\sqrt{d-1}))}$$

$$(30) \quad \geq \frac{\gamma^d(B(0, \frac{\sqrt{3}}{2}\sqrt{d-1}) \cap B(\sqrt{d-1} e_1, \frac{\sqrt{3}}{2}\sqrt{d-1}))}{\gamma^d(B(\sqrt{d-1} e_1, (\frac{\sqrt{3}}{2} + \frac{1}{d-1})\sqrt{d-1}))}$$

$$(31) \quad \geq \frac{\gamma^d(B(0, \frac{\sqrt{3}}{2}\sqrt{d-1}) \cap B(\sqrt{d-1} e_1, \frac{\sqrt{3}}{2}\sqrt{d-1}))}{V(\frac{\sqrt{3}}{2} + \frac{1}{d-1})}$$

$$(32) \quad = \frac{V(\frac{\sqrt{3}}{2})}{V(\frac{\sqrt{3}}{2} + \frac{1}{d-1})} \frac{\gamma^d(B(0, \frac{\sqrt{3}}{2}\sqrt{d-1}) \cap B(\sqrt{d-1} e_1, \frac{\sqrt{3}}{2}\sqrt{d-1}))}{V(\frac{\sqrt{3}}{2})}.$$

The last factor in the preceding line is bounded below (cf. [CriSjo, p. 610]) by

$$(33) \quad \frac{c_0}{\sqrt{d-1}},$$

where c_0 is a strictly positive constant (in particular, it is independent of d ; it may depend on the choices of R and r , which are equal to $\sqrt{3}/2$ in our case). Regarding

$$\frac{V(\frac{\sqrt{3}}{2})}{V(\frac{\sqrt{3}}{2} + \frac{1}{d-1})},$$

as d becomes large, the changes that R and R^2 undergo in the fraction of formula (28), when R takes the value $\frac{\sqrt{3}}{2} + \frac{1}{d-1}$ instead of $\frac{\sqrt{3}}{2}$, become vanishingly small, so all we need to do is to bound

$$(34) \quad \left(\frac{G\left(\frac{\sqrt{3}}{2}\right)}{G\left(\frac{\sqrt{3}}{2} + \frac{1}{d-1}\right)} \right)^{\frac{d-1}{2}}$$

from below. A computation shows that $G''(\sqrt{3}/2) < 0$, so G is locally concave at $\sqrt{3}/2$, and thus, for d sufficiently high, $G(\sqrt{3}/2) + 1/(d-1) \leq G(\sqrt{3}/2) + G'(\sqrt{3}/2)/(d-1)$. Since

$G(\sqrt{3}/2) = 1/(2e^{3/4})$ and $G'(\sqrt{3}/2) = \sqrt{3}/(2e^{3/4})$, we have that

$$(35) \quad \left(\frac{G\left(\frac{\sqrt{3}}{2}\right)}{G\left(\frac{\sqrt{3}}{2} + \frac{1}{d-1}\right)} \right)^{\frac{d-1}{2}} \geq \left(\frac{1}{1 + \frac{\sqrt{3}}{d-1}} \right)^{\frac{d-1}{2}} > e^{-1}$$

for d large enough.

Finally, since $g(r) := r^{d-1}e^{-r^2/2}$ is increasing for $0 < r < \sqrt{d-1}$,

$$\begin{aligned} \gamma^d(B(0, \frac{\sqrt{3}}{2}\sqrt{d-1})) &= \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{(2\pi)^{d/2}} \int_0^{\frac{\sqrt{3}}{2}\sqrt{d-1}} g(r) dr \\ &\leq \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{(2\pi)^{d/2}} g\left(\frac{\sqrt{3}}{2}\sqrt{d-1}\right) \int_0^{\frac{\sqrt{3}}{2}\sqrt{d-1}} dr = \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{(2\pi)^{d/2}} \left(\frac{\sqrt{3}}{2}\sqrt{d-1}\right)^d e^{-\frac{3d-3}{8}}. \end{aligned}$$

Using Stirling's formula, for d large the right hand side of the preceding equality can be bounded above by

$$\sqrt{d} \left(\frac{3e^{1/4}}{4} \right)^{\frac{d}{2}}.$$

Putting together in formula (25) the bounds (22)-(23), (27)-(35), and the last estimate, we conclude that for d sufficiently large,

$$\|A_{\sqrt{3d-3}/2}\|_{L^p \rightarrow L^{p,\infty}} \geq \left(\frac{2}{3^{1/2}e^{1/8}} \right)^{\frac{d}{p}} \Theta \left(\frac{1}{d^{1/2+1/p}} \right).$$

Now $\frac{2}{3^{1/2}e^{1/8}} > 1.019$, so for d large enough the factor $\Theta \left(\frac{1}{d^{1/2+1/p}} \right)$ can be absorbed in the exponential, and we get

$$\left\| A_{\frac{\sqrt{3d-3}}{2}} \right\|_{L^p \rightarrow L^{p,\infty}} > 1.019^{d/p}.$$

□

I do not know whether the uniform L^1 boundedness of the operators A_r in the two preceding cases (exponential and gaussian) are instances of a more general result in euclidean spaces, or whether there are measures ν in \mathbb{R}^d for which $\sup_{r>0} \|A_{r,\nu}\|_{L^1 \rightarrow L^1} = \infty$. Curiously, for the one-directional averaging operators in \mathbb{R} , examples of such measures are easy to find.

Given μ on \mathbb{R} and $s > 0$, define, for all $x \in \mathbb{R}$ such that $\mu([x, x+s]) > 0$, the right directional averaging operator as

$$A_{s,\mu}^r f(x) := \frac{1}{\mu([x, x+s])} \int_{\mathbb{R}} f(y) \mathbf{1}_{[x, x+s]}(y) d\mu(y).$$

Theorem 4.8. *There exists a measure ν on \mathbb{R} such that $\|A_{1,\nu}^r\|_{L^1 \rightarrow L^1} = \infty$.*

By way of comparison, for $p > 1$ and denoting by M^r the right directional Hardy-Littlewood maximal operator, given any μ on \mathbb{R} we have that

$$\sup_{s>0} \|A_{s,\mu}^r\|_{L^p \rightarrow L^p} \leq \|M_\mu^r\|_{L^p \rightarrow L^p} = \frac{p}{p-1}$$

(cf. [Gra, p. 102, Exercise 2.1.11]).

Proof. Let $B := \cup_{n \geq 0} [2n, 2n+1)$, and let $d\nu(x) := (\mathbf{1}_B(x) + e^{-x})dx$. Again by approximation, we can use Dirac deltas instead of functions. For $x \in [2n, 2n+1)$ we have that

$$A_{1,\nu}^r \delta_{2n+1}(x) \geq \frac{1}{2n+1-x+e^{-2n}}.$$

Since $d\nu(x) > 1$ on $[2n, 2n+1)$, we conclude that $\lim_{n \rightarrow \infty} \|A_{1,\nu}^r \delta_{2n+1}\|_{L^1} = \infty$. \square

5. LOCAL COMPARABILITY VS DOUBLING

Next we study the local comparability condition and its relationship to doubling. Recall that any such comparison must be made in geometrically doubling spaces, the only ones that support doubling measures.

Example 5.1. For a very simple example of a nondoubling locally comparable measure, on a geometrically doubling space, just take the space X of 2 points x and y at distance 1, and let $\mu := \delta_x$. Then balls are either disjoint or the whole space (when $r > 1$), so $C(\mu) = 1$. Actually, the same argument shows that $C(\nu) = 1$ for every measure ν on X . A variant of the preceding example, but admitting nondoubling measures with full support, is given by \mathbb{N} with $d(m, n) = 1$ when $m \neq n$. If μ is any finite Borel measure on \mathbb{N} , then $C(\mu) = 1$. Alternatively, one can just recall Example 3.2, noting that the preceding spaces are (discrete) ultrametric.

Example 5.1 shows that a nontrivial locally comparable measure, can assign measure zero to some balls. For another difference between doubling and local comparability, note that if μ is doubling and $\mu\{x\} > 0$, then x is an isolated point of X . This need not be the case when we only have local comparability, as can be seen by choosing a measure with atoms in an ultrametric space without isolated points.

Speaking loosely, the better the connectivity properties of the space, and the smaller the “gaps” or “holes” in it, the more similar to doubling are the measures satisfying local comparability. For instance, if X is connected, then the local comparability of μ entails that no ball has measure 0, and furthermore, μ is continuous, that is, for every $x \in X$, $\mu\{x\} = 0$. The next result does not require X to be geometrically doubling.

Theorem 5.2. *If a metric measure space (X, d, μ) is connected and μ satisfies a local comparability condition, then all balls have strictly positive measure, and for every $x \in X$, $\mu\{x\} = 0$.*

Proof. If for some $r > 0$ and some $x \in X$, $\mu B(x, r) = 0$, then by local comparability, for every $y \in B(x, r)$, $\mu B(y, r) = 0$, and by τ -smoothness, $\mu Bl(x, r, r) = 0$. Define $Bl_1 := Bl(x, r, r)$ and for $n \geq 1$, $Bl_{n+1} := Bl(Bl_n, r)$. Again by τ -smoothness, for all $n \geq 1$, $\mu Bl_n = 0$, so $\mu \cup_n Bl_n = 0$. But $\cup_n Bl_n = X$ (this is well known and it follows from the fact that $\cup_n Bl_n$ is nonempty, open, and closed, so it is X ; in topological terminology, connected metric spaces are chainable). Thus, the nontriviality of μ is contradicted.

Next, suppose that for some $x \in X$ we have $\mu\{x\} > 0$. Select $n \in \mathbb{N} \setminus \{0\}$ and $r > 0$ such that $\mu(B(x, r)) < (1+1/n)\mu\{x\}$. Since x is not isolated there exists a $y \in B(x, r/3) \setminus \{x\}$. Now

if $B(x, d(x, y)) \cap B(y, d(x, y)) \neq \emptyset$, we can use these two balls to conclude that a local comparability condition cannot hold, since $\mu B(x, d(x, y)) / \mu B(y, d(x, y)) \geq \mu\{x\} / \mu B(y, d(x, y)) > n$ for n arbitrary. So assume that $B(x, d(x, y)) \cap B(y, d(x, y)) = \emptyset$. By connectivity, the closed ball $B^{cl}(x, d(x, y))$ is not open, whence there exists a $z \in B^{cl}(x, d(x, y))$ and a sequence $\{z_n\}_1^\infty$ in $(B^{cl}(x, d(x, y)))^c$ such that $\lim_n d(z_n, z) = 0$. Select $z_N \in B(z, d(x, y)/3)$. Then $d(x, z_N) > d(x, z) = d(x, y)$ and $2d(x, z_N) < r$, so $z \in B(x, d(x, z_N)) \cap B(z_N, d(x, z_N))$ and as before, $\mu B(x, d(x, z_N)) / \mu B(z_N, d(x, z_N)) > n$. \square

Lemma 5.3. *Let (X, d, μ) be a metric measure space, D -geometrically doubling, and such that μ satisfies a local comparability condition. Then $\mu(Bl(x, r, r)) \leq D C_\mu^3 \mu(B(x, r))$, and $\mu(Blu(x, r, r)) \leq D^2 C_\mu^4 \mu(B(x, r))$.*

Proof. Cover the ball $B(x, 2r)$ with at most D balls of radius r , and disregard all such balls having empty intersection with $Bl(x, r, r)$. Since $Bl(x, r, r) \subset B(x, 2r)$, this yields a cover $B(y_1, r), \dots, B(y_M, r)$ of $Bl(x, r, r)$ with $M \leq D$, and by Lemma 3.3, such that for every $1 \leq k \leq M$, $\mu B(y_k, r) \leq C_\mu^3 \mu B(x, r)$. Hence $\mu Bl(x, r, r) \leq D C_\mu^3 \mu(B(x, r))$.

The second inequality is proven in the same way: $Blu(x, r, r) \subset B(x, 3r)$; cover $B(x, 3r)$ with at most D^2 balls of radius r , and disregard all such balls having empty intersection with $Blu(x, r, r)$. This yields a cover $B(y_1, r), \dots, B(y_M, r)$ of $Blu(x, r, r)$ with $M \leq D^2$, and (applying Lemma 3.3 twice) such that for every $1 \leq k \leq M$, $\mu B(y_k, r) \leq C_\mu^4 \mu B(x, r)$. Hence $\mu Blu(x, r, r) \leq D^2 C_\mu^4 \mu B(x, r)$. \square

It is obvious that in an arbitrary metric measure space, boundedness of blossoms entails local comparability, since whenever $d(x, y) < r$, $B(x, r) \subset Bl(y, r, r)$ and $B(y, r) \subset Bl(x, r, r)$. If additionally the space is geometrically doubling, then the conditions are equivalent.

Corollary 5.4. *Let (X, d, μ) be a geometrically doubling metric measure space. The following are equivalent:*

- a) μ satisfies a local comparability condition.
- b) There exists a constant $K_1 \geq 1$ such that for every $x \in X$ and every $r > 0$, $\mu(Bl(x, r, r)) \leq K_1 \mu(B(x, r))$.
- c) There exists a constant $K_2 \geq 1$ such that for every $x \in X$ and every $r > 0$, $\mu(Blu(x, r, r)) \leq K_2 \mu(B(x, r))$.

In a certain sense, it could be said that from the viewpoint of the maximal operator the doubling condition is irrelevant, since in geometrically doubling spaces it can be replaced by local comparability (by the preceding corollary together with the Vitali covering lemma) and off geometrically doubling spaces, there are no doubling measures. However, if nearby points can always be joined by bounded chains of intersecting balls with the same radius, then in a geometrically doubling space, local comparability implies doubling. Hence, in many spaces of interest both conditions are equivalent.

Lemma 5.5. *Let (X, d, μ) be a metric measure space, D -geometrically doubling, and such that μ satisfies a local comparability condition. Suppose there exists a $K > 1$ such that for all $x, z \in X$ and all $r > 0$, whenever $d(x, z) < 2r$ there exists a chain of balls $B(x, r) =$*

$B(y_0, r), \dots, B(y_m, r)$ satisfying $m \leq K$, $z \in B(y_m, r)$ and for $j = 0, \dots, m-1$, $B(y_j, r) \cap B(y_{j+1}, r) \neq \emptyset$. Then μ is $D C_\mu^{2K+3}$ -doubling.

Proof. Cover $B(x, 2r)$ with at most D balls of radius r . Of course, any ball that does not intersect $B(x, 2r)$ can be disregarded, so suppose $B(w, r)$ is one of the balls in the cover, and let $z \in B(w, r) \cap B(x, 2r)$. Since $d(x, z) < 2r$, there is an intersecting chain of balls $B(x, r) = B(y_1, r), \dots, B(y_m, r)$ such that $m \leq K$ and $z \in B(y_m, r)$. By repeated application of Lemma 3.3, together with the fact that $d(y_m, z) < r$, we conclude that $\mu(B(z, r)) \leq C_\mu^{2K+1} \mu(B(x, r))$, so $\mu(B(w, r)) \leq C_\mu^{2K+3} \mu(B(x, r))$, and the result follows. \square

Next we indicate some conditions ensuring that the hypothesis of the previous lemma holds.

Definition 5.6. A metric space has the *approximate midpoint property* if for every $\varepsilon > 0$ and every pair of points x, y , there exists a point z such that $d(x, z), d(z, y) < \varepsilon + d(x, y)/2$.

Definition 5.7. A metric space is *quasiconvex* if there exists a constant $C \geq 1$ such that for every pair of points x, y , there exists a curve with x and y as endpoints, such that its length is bounded above by $Cd(x, y)$.

We say that $B(y_0, r), \dots, B(y_m, r)$ form an intersecting chain of balls if for $j = 0, \dots, m-1$, $B(y_j, r) \cap B(y_{j+1}, r) \neq \emptyset$.

Corollary 5.8. Let (X, d, μ) be a metric measure space, D -geometrically doubling, and such that μ satisfies a local comparability condition. If either all balls are connected, or X is quasiconvex, or it has the approximate midpoint property, then μ is doubling.

Proof. Whenever we have a cover, we assume that all sets in it intersect the set to be covered; otherwise, we disregard those not satisfying the condition.

Fix $B_0 := B(x, r)$. Suppose first that all the balls of X are connected. Let the collection $\mathcal{C} := \{B(x_1, r), \dots, B(x_M, r)\}$ be a cover of $B(x, 2r)$ with $M \leq D$, and write $B_i := B(x_i, r)$. Let \mathcal{C}' be the collection of all balls B_i in \mathcal{C} for which there is an intersecting chain of balls starting at B_0 and finishing with B_i . Then $\mathcal{C} = \mathcal{C}'$, for otherwise the union of all balls in \mathcal{C}' and the union of all balls in $\mathcal{C} \setminus \mathcal{C}'$ would form a disconnection of $B(x, 2r)$. Adding B_0 to the balls in the cover, we see that the maximal length of any chain is $M + 1 \leq D + 1$.

Suppose next that X is quasiconvex with constant C_X (any constant satisfying Definition 5.7; there might not be a smallest one). Choose y with $d(x, y) < 2r$. Then there exists a curve $c : [0, 1] \rightarrow X$ starting at x and finishing at y (that is, $c(0) = x$, $c(1) = y$) such that its length $L(c)$ satisfies $L(c) < C_X 2r$. Let $K = \lfloor L(c)/r \rfloor + 1$, where $\lfloor L(c)/r \rfloor$ denotes the integer part of $L(c)/r$. Divide c into K subsegments of equal length, with endpoints $x_0 = x, x_1, \dots, x_{K-1}, x_K = y$. Then the balls $\{B(x_0, r), \dots, B(x_{K-1}, r)\}$ form an intersecting chain of length $K \leq \lfloor 2C_X \rfloor + 1$.

The argument for the case where X has the approximate midpoint property is similar and simpler, so we omit it. \square

If none of the conditions in the preceding result hold, the equivalence between local comparability and doubling can fail, even for arc-connected spaces.

Theorem 5.9. *There exists an arc-connected, geometrically doubling metric measure space (X, d, μ) , such that $X \subset \mathbb{R}^2$, d is defined by the restriction of the ℓ_∞ -norm to X , and μ satisfies a local comparability condition but is not doubling.*

Proof. On \mathbb{R} set $d\nu(x) = dx$ for $x \leq 1$, and $d\nu(x) = xdx$ for $x \geq 1$ (we mention that ν does not satisfy a local comparability condition, since $\lim_{x \rightarrow \infty} \nu([0, x])/\nu([-x, 0]) = \infty$, so in particular it is not doubling). Next we define an embedding $f : \mathbb{R} \rightarrow \mathbb{R}^2$ as follows: $f : (-\infty, -1] \rightarrow [0, \infty) \times \{0\}$ is given by $f(t) = (-1 - t, 0)$, $f : [-1, 0] \rightarrow \{0\} \times [0, 1]$ is given by $f(t) = (0, 1 + t)$, and $f : [0, \infty) \rightarrow [0, \infty) \times \{1\}$, by $f(t) = (t, 1)$. Then we set $X := f(\mathbb{R}) \subset \mathbb{R}^2$, with d on X defined by the ℓ_∞ norm on the plane: $d((a, b), (c, d)) = \max\{|a - c|, |b - d|\}$. Let μ be the pushforward measure $f_*\nu$, so $\mu A := \nu(f^{-1}(A))$. Then $\lim_{x \rightarrow \infty} \mu(B((x, 0), 2))/\mu(B((x, 0), 1)) = \infty$, so μ is not doubling. However, since $B((x, 0), t) = B((x, 1), t)$ for $t > 1$, while $B((x, 0), t) \cap B((x, 1), t) = \emptyset$ for $0 < t \leq 1$ and $x \geq 1$, it is not difficult to see that μ satisfies a local comparability condition. More precisely, let $S := f(-\infty, 1]$ and note that on S , μ is just length. Thus, for balls centered at points $x = (t, 0)$ or $x = (0, t)$ with $r \leq 1$, the result is clear, while if $r > 1$, then $x = (t, 1)$ is also a center of $B((t, 0), r)$, so it is enough to consider the case $x = (t, 1)$, $t \geq 0$. Since

$$\mu(S \cap B((t, 1), r)) \geq \mu(S \cap B((t + r, 1), r)),$$

we have

$$\begin{aligned} \frac{\mu(B((t + r, 1), r))}{\mu(B((t, 1), r))} &= \frac{\mu(S \cap B((t + r, 1), r)) + \mu(S^c \cap B((t + r, 1), r))}{\mu(S \cap B((t, 1), r)) + \mu(S^c \cap B((t, 1), r))} \\ &\leq \frac{\mu(S^c \cap B((t + r, 1), r))}{\mu(S^c \cap B((t, 1), r))} \leq \frac{\int_t^{t+2r} u du}{\int_t^{t+r} u du} \leq 4. \end{aligned}$$

(A more involved argument yields the optimal constant $C(\mu) = 2$). \square

Note that in the space $X = f(\mathbb{R})$ defined above, if $x \geq 2$, then $Blu((x, 0), 1, 1)$ does not contain any ball strictly larger than $B((x, 0), 1)$; in particular, for all $t > 0$, $B((x, 0), 1 + t) \not\subset Blu((x, 0), 1, 1)$. So even in arc-connected spaces, blossoms and balls can be rather different. On the other hand, if there exists a fixed $t > 0$ such that for every $x \in X$ and every $r > 0$, $B(x, (1 + t)r) \subset Bl(x, r, r)$, then boundedness of blossoms entails doubling.

6. REMARKS ON SPACES THAT MAY FAIL TO BE GEOMETRICALLY DOUBLING

Boundedness of blossoms does not imply that X is geometrically doubling.

Theorem 6.1. *There exists a metric measure space (X, d, μ) , such that (X, d) is not geometrically doubling, μ satisfies a local comparability condition, and blossoms boundedly. Furthermore, μ can be chosen to have full support.*

Proof. We use the infinite broom $B \subset \mathbb{R}^2$, $B := \cup_{n \in \mathbb{N}} \{(x, nx) : x \geq 0\}$, with d the path metric. Now for $n \geq 1$, let $z_n \in \{(x, nx) : x \geq 0\}$ be the only point that satisfies $d(0, z_n) = 1/n$. Set $X := \{0\} \cup \{z_n : n \geq 1\} \subset B$, with the distance inherited from B . To see that (X, d) is not geometrically doubling, note that $B(0, 1/n)$ contains the following disjoint

balls: $B(0, 1/(2n))$, $B(z_{n+1}, 1/(2n))$, \dots , $B(z_{2n}, 1/(2n))$. Next, let $\mu = \delta_0$ be the point mass at the origin. If two balls B_1, B_2 centered at points of X intersect, they both must contain 0, so $\mu B_1 = \mu B_2 = 1$, and hence $C(\mu) = 1$. Blossoming at 0 does not change the mass, while blossoming at other points leads to either not increasing the original ball, or not increasing its measure: If $0 \notin B(z_k, r)$, then $Blu(z_k, r, r) = B(z_k, r)$, and if $0 \in B(z_k, r)$, then $\mu Blu(z_k, r, r) = \mu B(z_k, r) = 1$.

One can easily modify μ so that it also has full support: In addition to the Dirac delta at the origin, give mass 2^{-n} to each z_n , and argue essentially as before. \square

Beyond geometrically doubling metric spaces, it is unclear to me whether or not local comparability suffices to obtain boundedness results for the Hardy-Littlewood maximal operator. Of course, if blossoms are bounded, then the usual Vitali covering argument works. Nevertheless, in specific and natural examples, such as volume in hyperbolic spaces (cf. [Str], [LiLo]), blossoms are not bounded, and still weak type $(1, 1)$ bounds hold for the centered maximal operator (the uncentered operator is in general unbounded, as can be seen by considering just one Dirac delta in the hyperbolic plane). A rather different instance of this phenomenon is presented in [NaTa, Theorem 1.5], where boundedness of the maximal operator is obtained for the infinite, rooted k -ary tree with the standard graph metric. In these examples, however, one not only has local, but global comparability with constant 1, since the measure of balls only depends on the radius, and not the center.

While writing this paper I found [SoTr], where local comparability is considered, under the name of “equidistant comparability property”, in the specific case of connected graphs with all vertices having finite degree. Here X is the set of vertices V of a graph, $d(x, y)$ is defined as the smallest number of edges one needs to traverse in order to go from x to y , and μ is the counting measure. Since all vertices have finite degree, the measure of all balls is finite, and we are within the general framework considered in the present paper. For instance, it is easy to check that for the metric measure spaces considered here, the centered maximal operator acting on just one Dirac delta is weak $(1, 1)$ bounded, with bound C_μ (this is so even if the Dirac delta is placed at a point outside the support of μ). The special case of this result for graphs appears in [SoTr, Proposition 2.10].

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